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Accidental degeneracies and symmetry group of the harmonic oscillator in a strong magnetic field

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Abstract. The problem of an electron in a general central potential, subject to a constant external magnetic field, is known to have no degeneracy and to be invariant under SO(2). However, once the central potential has a symmetry group larger than SO(3), even in the presence of a magnetic field, residual accidental degeneracies may remain and a symmetry group larger than SO(2) may exist. In the present paper, we investigate the case where the central potential is a harmonic oscillator and the magnetic field is strong, meaning that the Hamiltonian contains both a linear and a quadratic term in the magnetic field intensity $\mathscr{H}$. In a recent work, Moshinsky *et al* did claim that in such a case no accidental degeneracy is left. It is shown here that, although this assertion is valid for most values of $\mathscr{H}$, there exist values for which there are residual accidental degeneracies. The analysis is based on a linear canonical transformation converting the Hamiltonian into that of an anisotropic oscillator in the absence of magnetic field. It is shown that the accidental degeneracies of the latter are either due to an SU(2), an SU(3) or an SU(2)×SU(2) symmetry group, or cannot be explained by the existence of any symmetry group. In particular, it is proved that accidental degeneracies, and a corresponding SU(2) symmetry group, may appear when the frequencies are irrationally related. Finally the accidental degeneracies and the associated symmetry group of the isotropic oscillator in a strong magnetic field (when it exists) are determined in terms of the ratio between the oscillator and cyclotron frequencies.

1. Introduction

In the spectra of some quantum mechanical Hamiltonians, there appear 'accidental' degeneracies, namely degeneracies larger than those to be expected from the obvious geometrical symmetries of the Hamiltonian, such as invariance under rotations in the case of central potentials. Two well known examples of such Hamiltonians are those of the Coulomb and harmonic oscillator. Some fifty years ago, it was shown by Fock (1935), Bargmann (1936) and Jauch and Hill (1940) that the accidental degeneracies occurring in those two cases are due to a larger symmetry group than the SO(3) group connected with the rotational invariance, namely SO(4) for the Coulomb Hamiltonian and SU(3) for the harmonic oscillator Hamiltonian.

Ever since, much work has been devoted to the search for a larger symmetry group responsible for the accidental degeneracies whenever the latter do occur (McIntosh 1971, Louck *et al* 1973a, b, Moshinsky *et al* 1975, Moshinsky and Patera 1975). Despite these achievements, it has become clear that accidental degeneracy is not always connected with the existence of a symmetry group (Cisneros and McIntosh 1970, Louck and Metropolis 1981, Moshinsky 1983, Moshinsky and Quesne 1983).

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In the present paper, we consider the problem of an electron in an isotropic harmonic oscillator potential plus a constant external magnetic field. On the assumption that the latter is strong, the Hamiltonian contains both a linear and a quadratic term in the magnetic field intensity $\mathscr{H}$. Apart from its interest to astrophysics, where magnetic fields of the order of 10^{12} G are observed in the vicinity of pulsars (Smith 1977), that problem can also be useful in some earthly applications. Such is the case in the study of size effects on the diamagnetic properties of small metallic particles, wherein the harmonic oscillator serves to confine the electrons within the small particles (Denton 1973). To obtain the diamagnetic susceptibility, one indeed needs to calculate the level spectrum to the order of $\mathscr{H}^2$.

The problem of a charged particle in a general central potential subject to a constant external magnetic field is known to have no degeneracy and to be invariant under SO(2). However, once the central potential has a symmetry group larger than SO(3), even in the presence of a magnetic field, residual accidental degeneracies may remain, and a symmetry group larger than SO(2) may exist. In a recent work, Moshinsky *et al* (1984) discussed the case where the central potential is an isotropic harmonic oscillator. They considered both the strong field case, and the weak or very strong field limits, wherein either only the linear term or the quadratic term in $\mathscr{H}$ has to be taken into account. In the strong field case, they claim that there is no accidental degeneracy left, and the Hamiltonian symmetry group therefore reduces to SO(2).

The purpose of the present paper is to show that although the assertion of Moshinsky *et al* is valid for most values of $\mathscr{H}$, there do exist values for which there are residual accidental degeneracies and consequently a more extensive symmetry group than SO(2) may be expected. A search for the latter will also be carried out and will lead to the conclusion that the accidental degeneracies cannot always be explained by the existence of a symmetry group.

Instead of the cylindrical or spherical coordinates used by Moshinsky *et al*, we shall carry out our analysis in cartesian coordinates. By means of a linear canonical transformation we shall reduce the problem to that of an anisotropic oscillator in the absence of a magnetic field. In § 2, we begin by solving the two-dimensional problem and then outline the analysis of the three-dimensional one. We discuss the accidental degeneracies of the three-dimensional anisotropic oscillator in § 3, and search for the corresponding symmetry group in § 4. Finally, in § 5, we apply the results of the previous sections to the three-dimensional isotropic oscillator in a strong magnetic field.

2. The solution of the two-dimensional problem and its extension to three dimensions

Let us consider a non-relativistic electron of mass m and charge $-e$, moving in a two-dimensional isotropic oscillator potential of frequency Ω in the x_1x_2 plane, and subjected to a constant external magnetic field $\boldsymbol{\mathscr{H}}$ along the x_3 axis. In terms of the coordinates and momenta in standard units x_i'', p_i'', $i=1,2$, the Hamiltonian of the electron reads

$$H''=(1/2m)\sum_{i=1}^{2}[p_i''+(e/c)A_i'']^2+\tfrac{1}{2}m\Omega^2\sum_{i=1}^{2}(x_i'')^2 \tag{2.1}$$

where in the symmetrical gauge

$$\boldsymbol{A}''=\tfrac{1}{2}\boldsymbol{\mathscr{H}}\times\boldsymbol{x}''. \tag{2.2}$$

If we denote by x'_i, p'_i, H' the coordinates, momenta and Hamiltonian in atomic units, i.e.

$$x'_i = (me^2/\hbar^2)x''_i \qquad p'_i = (\hbar/me^2)p''_i \qquad H' = (\hbar^2/me^4)H'' \tag{2.3}$$

equation (2.1) becomes

$$H' = \tfrac{1}{2}\sum_{i=1}^{2}[p_i'^2 + (b^4+\varepsilon^4)x_i'^2] + b^2(x'_1p'_2 - x'_2p'_1). \tag{2.4}$$

Here b^2 and ε^2 are dimensionless constants, respectively defined by

$$b^2 = \hbar^3\mathscr{H}/2m^2ce^3 \qquad \text{and} \qquad \varepsilon^2 = \hbar^3\Omega/me^4. \tag{2.5}$$

The Hamiltonian H', being quadratic in the coordinates and momenta, can be transformed via the linear canonical transformation (Dulock and McIntosh 1966, Moshinsky and Winternitz 1980, Schuh 1985)

$$\begin{aligned} x'_1 &= (b^4+\varepsilon^4)^{-1/4}(x_1-p_2)/\sqrt{2} \qquad & p'_1 &= (b^4+\varepsilon^4)^{1/4}(x_2+p_1)/\sqrt{2} \\ x'_2 &= (b^4+\varepsilon^4)^{-1/4}(x_2-p_1)/\sqrt{2} \qquad & p'_2 &= (b^4+\varepsilon^4)^{1/4}(x_1+p_2)/\sqrt{2} \end{aligned} \tag{2.6}$$

into the Hamiltonian of a two-dimensional anisotropic oscillator in appropriate units

$$H = (b^4+\varepsilon^4)^{-1/2}H' = \sum_{i=1}^{2}\omega_i\tfrac{1}{2}(p_i^2+x_i^2). \tag{2.7}$$

The frequencies of the latter are given by

$$\omega_1 = 1+\alpha \qquad \omega_2 = 1-\alpha \tag{2.8}$$

where α is a dimensionless parameter defined by

$$\alpha = b^2(b^4+\varepsilon^4)^{-1/2}. \tag{2.9}$$

In terms of the cyclotron frequency $\omega_c = e\mathscr{H}/mc$, α can be rewritten as

$$\alpha = (1+4\beta^2)^{-1/2} \qquad \text{where} \qquad \beta = \Omega/\omega_c \tag{2.10}$$

and takes values in the interval (0, 1), the weak and very strong field limits corresponding respectively to values close to 0 or 1.

In terms of the creation and annihilation operators η_i and ξ_i, $i = 1, 2$, defined by

$$\eta_i = (x_i - ip_i)/\sqrt{2} \qquad \xi_i = (x_i + ip_i)/\sqrt{2} \tag{2.11}$$

and satisfying the boson commutation relations

$$[\xi_i, \eta_j] = \delta_{ij} \qquad [\eta_i, \eta_j] = [\xi_i, \xi_j] = 0 \tag{2.12}$$

the Hamiltonian H can be written as

$$H = \sum_{i=1}^{2}(\eta_i\xi_i + \tfrac{1}{2})\omega_i. \tag{2.13}$$

Its eigenvalues (and therefore also those of H'' in convenient units) are given by

$$E_{\nu_1\nu_2} = \sum_{i=1}^{2}(\nu_i+\tfrac{1}{2})\omega_i \qquad \nu_1, \nu_2 = 0, 1, 2, \ldots \tag{2.14}$$

and the corresponding eigenstates are

$$|\nu_1\nu_2\rangle = \prod_{i=1}^{2} (\nu_i!)^{-1/2}\eta_i^{\nu_i}|0\rangle \tag{2.15}$$

where $|0\rangle$ denotes the boson vacuum state.

It is well known (Cisneros and McIntosh 1970, Louck *et al* 1973a, Moshinsky *et al* 1975) that accidental degeneracies occur in the spectrum of H if and only if the frequency ratio is rational, i.e.

$$\omega_2/\omega_1 = k_1/k_2 \qquad \text{or} \qquad k_1\omega_1 = k_2\omega_2 = \omega \tag{2.16}$$

where k_1 and k_2 are two relatively prime integers. In this case, H has an SU(2) symmetry group whose Lie algebra has been constructed by Louck *et al* (1973a). We shall now review their analysis with the purpose of extending it to three dimensions in the subsequent sections.

One begins by dividing the set of states (2.15) into k_1k_2 subsets characterised by a pair of indices $\boldsymbol{\lambda} = (\lambda_1\lambda_2)$ defined by

$$\nu_i = \lambda_i \bmod k_i \qquad \lambda_i = 0, 1, \ldots, k_i - 1 \qquad i = 1, 2. \tag{2.17}$$

The states belonging to a given subset $\boldsymbol{\lambda}$ are labelled by two non-negative integers n_1, n_2 as follows:

$$|n_1n_2)_{\boldsymbol{\lambda}} = |n_1k_1 + \lambda_1, n_2k_2 + \lambda_2\rangle \tag{2.18}$$

and span a subspace $S^{(\boldsymbol{\lambda})}$ of Hilbert space S. In terms of n_1, n_2, the eigenvalues (2.14) can be rewritten as

$$E_n^{(\boldsymbol{\lambda})} = \left(n + \sum_{i=1}^{2} (\lambda_i + \tfrac{1}{2})k_i^{-1}\right)\omega \qquad n = \sum_{i=1}^{2} n_i. \tag{2.19}$$

From equation (2.19), it follows that those members of subset $\boldsymbol{\lambda}$, for which $n = n_1 + n_2$ is the same, are degenerate. Hence each subset $\boldsymbol{\lambda}$ can be put into one-to-one correspondence with the full set of eigenstates of a two-dimensional isotropic oscillator.

In each subspace $S^{(\boldsymbol{\lambda})}$, one next defines new operators $\bar{\eta}_i^{(\boldsymbol{\lambda})}$ and $\bar{\xi}_i^{(\boldsymbol{\lambda})}$, $i = 1, 2$, by the following relations:

$$\begin{aligned} \bar{\eta}_i^{(\boldsymbol{\lambda})} &= k_i^{-1/2}(\eta_i\xi_i - \lambda_i)^{1/2}[\eta_i\xi_i(\eta_i\xi_i - 1)\ldots(\eta_i\xi_i - k_i + 1)]^{-1/2}\eta_i^{k_i} \\ \bar{\xi}_i^{(\boldsymbol{\lambda})} &= k_i^{-1/2}\xi_i^{k_i}[\eta_i\xi_i(\eta_i\xi_i - 1)\ldots(\eta_i\xi_i - k_i + 1)]^{-1/2}(\eta_i\xi_i - \lambda_i)^{1/2}. \end{aligned} \tag{2.20}$$

When applied to the states (2.18), these new operators behave as usual creation and annihilation operators, since from equation (2.20), one obtains

$$\begin{aligned} \bar{\eta}_1^{(\boldsymbol{\lambda})}|n_1n_2)_{\boldsymbol{\lambda}} &= (n_1 + 1)^{1/2}|n_1 + 1, n_2)_{\boldsymbol{\lambda}} \\ \bar{\xi}_1^{(\boldsymbol{\lambda})}|n_1n_2)_{\boldsymbol{\lambda}} &= n_1^{1/2}|n_1 - 1, n_2)_{\boldsymbol{\lambda}} \end{aligned} \tag{2.21}$$

and similar relations for $\bar{\eta}_2^{(\boldsymbol{\lambda})}$ and $\bar{\xi}_2^{(\boldsymbol{\lambda})}$. In $S^{(\boldsymbol{\lambda})}$, $\bar{\eta}_i^{(\boldsymbol{\lambda})}$ and $\bar{\xi}_i^{(\boldsymbol{\lambda})}$, $i = 1, 2$, therefore satisfy commutation relations of the type (2.12). Under transformation (2.20), the restriction $H^{(\boldsymbol{\lambda})}$ of H to $S^{(\boldsymbol{\lambda})}$ is converted into a two-dimensional isotropic oscillator of frequency ω

$$H^{(\boldsymbol{\lambda})} = \left(\sum_{i=1}^{2} \bar{\eta}_i^{(\boldsymbol{\lambda})}\bar{\xi}_i^{(\boldsymbol{\lambda})} + \sum_{i=1}^{2} (\lambda_i + \tfrac{1}{2})k_i^{-1}\right)\omega. \tag{2.22}$$

Hence it has an SU(2) symmetry group, whose generators are the traceless operators obtained from the operators

$$C_{ij}^{(\lambda)} = \bar{\eta}_i^{(\lambda)} \bar{\xi}_j^{(\lambda)} \qquad i, j = 1, 2 \tag{2.23}$$

satisfying the U(2) commutation relations

$$[C_{ij}^{(\lambda)}, C_{kl}^{(\lambda)}] = \delta_{jk} C_{il}^{(\lambda)} - \delta_{il} C_{kj}^{(\lambda)}. \tag{2.24}$$

In Hilbert space S, one finally introduces projection operators $P^{(\lambda)}$ onto the various subspaces $S^{(\lambda)}$. Since $P^{(\lambda)}$ commutes with H, $\bar{\eta}_i^{(\lambda')}$, $\bar{\xi}_i^{(\lambda')}$, the operators

$$C_{ij} = \sum_{\lambda} C_{ij}^{(\lambda)} P^{(\lambda)} \tag{2.25}$$

leave H invariant, connect all its degenerate eigenstates, and satisfy the U(2) commutation relations (2.24). The corresponding traceless operators are therefore the generators of the SU(2) symmetry group of H.

Going back to the Hamiltonian H'' of the two-dimensional isotropic oscillator in a strong magnetic field, we conclude from the above analysis that it does have accidental degeneracies, and an associated SU(2) symmetry group for those values of the magnetic field intensity $\mathscr{H}$ for which the parameter α happens to be rational, meaning that β may be written in the form

$$\beta = \tfrac{1}{2}[(q/p)^2 - 1]^{1/2} \qquad p, q \in \boldsymbol{N}^+ \qquad p < q. \tag{2.26}$$

The analysis of the two-dimensional problem can easily be extended to three dimensions. The Hamiltonian H'' of the three-dimensional problem is given by equation (2.1), where the summation over i now goes from 1 to 3. It only differs from the Hamiltonian of the two-dimensional problem by the addition of a harmonic oscillator of frequency Ω in the x_3 direction, and in atomic units can be written as

$$H' = \tfrac{1}{2} \sum_{i=1}^{2} [p_i'^2 + (b^4 + \varepsilon^4) x_i'^2] + b^2 (x_1' p_2' - x_2' p_1') + \tfrac{1}{2}(p_3'^2 + \varepsilon^4 x_3'^2). \tag{2.27}$$

When one performs the linear canonical transformation defined by equation (2.6) and by the following relations

$$x_3' = \varepsilon^{-1} x_3 \qquad p_3' = \varepsilon p_3 \tag{2.28}$$

H' is converted into the Hamiltonian of a three-dimensional anisotropic oscillator, given by equation (2.7) where i now goes from 1 to 3, and the frequencies are defined by

$$\omega_1 = 1 + \alpha \qquad \omega_2 = 1 - \alpha \qquad \omega_3 = (1 - \alpha^2)^{1/2}. \tag{2.29}$$

To our knowledge, a thorough discussion of the accidental degeneracies and symmetry group of the three-dimensional anisotropic oscillator along the lines of the Louck *et al* analysis for two dimensions has not been carried out so far although some partial results are known (Cisneros and McIntosh 1970) and different analyses are available (Major 1977). We shall therefore proceed to fill this gap in §§ 3 and 4 before going back to the problem of an oscillator in a strong magnetic field in § 5.

3. Accidental degeneracies of the three-dimensional anisotropic oscillator

Let us consider the three-dimensional anisotropic oscillator, whose Hamiltonian is given by

$$H=\sum_{i=1}^{3}\omega_i\tfrac{1}{2}(p_i^2+x_i^2)=\sum_{i=1}^{3}(\eta_i\xi_i+\tfrac{1}{2})\omega_i \tag{3.1}$$

where η_i and ξ_i are defined in equation (2.11). Its eigenstates can be written as

$$|\nu_1\nu_2\nu_3\rangle=\prod_{i=1}^{3}(\nu_i!)^{-1/2}\eta_i^{\nu_i}|0\rangle \tag{3.2}$$

and correspond to the eigenvalues

$$E_{\nu_1\nu_2\nu_3}=\sum_{i=1}^{3}(\nu_i+\tfrac{1}{2})\omega_i \qquad \nu_1,\nu_2,\nu_3=0,1,2,\ldots. \tag{3.3}$$

There occur accidental degeneracies in the spectrum if and only if the three frequencies ω_i, $i=1,2,3$ satisfy some relation of the type

$$\sum_{i=1}^{3}\mu_i\omega_i=0 \qquad \mu_1,\mu_2,\mu_3\in \boldsymbol{Z}. \tag{3.4}$$

Without loss of generality, we may assume that $(\mu_1\mu_2\mu_3)$ is a set of three relatively prime integers. Contrary to what happens in the two-dimensional case, equation (3.4) may be fulfilled by either one or two different sets of relatively prime integers $(\mu_1\mu_2\mu_3)$. As shown in § 4, the resulting accidental degeneracy patterns and symmetry groups will then be different. In the remainder of this section, we shall determine the restrictions imposed on the frequency ratios ω_2/ω_1, ω_3/ω_1, ω_3/ω_2 by the existence of one or two independent relations of the type (3.4). As a preliminary remark, we note that since there are only two independent ratios, then if two of them are rational, so is the third.

Let us start with the case where equation (3.4) is satisfied by two different sets of relatively prime integers $(\mu_1\mu_2\mu_3)$ and $(\mu_1'\mu_2'\mu_3')$. Then μ_3 and μ_3' cannot simultaneously be equal to zero since ω_1 and ω_2 cannot satisfy two independent relations with relatively prime integer coefficients. Without loss of generality, we may assume that μ_3 is different from zero. If $\mu_3'=0$, then μ_1', $\mu_2'\neq 0$, and the ratio $\omega_2/\omega_1=-\mu_1'/\mu_2'$ is rational. If on the contrary $\mu_3'\neq 0$, then we can eliminate ω_3 between the two relations and write

$$(\mu_1\mu_3'-\mu_3\mu_1')\omega_1+(\mu_2\mu_3'-\mu_3\mu_2')\omega_2=0. \tag{3.5}$$

This condition imposes that either $\mu_1\mu_3'-\mu_3\mu_1'=\mu_2\mu_3'-\mu_3\mu_2'=0$, or ω_2/ω_1 is rational. Since the first alternative would imply that the two sets $(\mu_1\mu_2\mu_3)$ and $(\mu_1'\mu_2'\mu_3')$ are identical, we again find that ω_2/ω_1 must be rational. It can be proved in the same way that ω_3/ω_1 is also rational. We conclude that when equation (3.4) is satisfied by two different sets of relatively prime integers, the three frequency ratios are rational, i.e. $k_1\omega_1=k_2\omega_2=k_3\omega_3$ where k_1, k_2, k_3 are three relatively prime integers. The converse of this proposition obviously holds too. Note that if k_1, k_2, k_3 are relatively prime integers, they are not necessarily relatively prime per couple. For instance, $k_1=1$, $k_2=2$ and $k_3=2$ are relatively prime, but $k_2=2$ and $k_3=2$ are not. This will complicate the discussion in § 4.1 to a large extent.

Let us next consider the case where equation (3.4) is satisfied by only one set of relatively prime integers. From the above argument, it follows that no two frequency ratios can simultaneously be rational. Hence we have to examine the following two cases: either with one rational frequency ratio or without. In the former case, equation

(3.4) is automatically satisfied by one set of relatively prime integers $(\mu_1\mu_2\mu_3)$, e.g. if $\omega_2/\omega_1 = k_1/k_2$, then $\mu_1 = k_1$, $\mu_2 = -k_2$ and $\mu_3 = 0$ fulfil equation (3.4). In the latter case, equation (3.4) is not automatically satisfied but there do exist irrational values of the three frequency ratios for which it is fulfilled, e.g. if $\omega_2/\omega_1 = \sqrt{2}$, $\omega_3/\omega_1 = 1+\sqrt{2}$ and $\omega_3/\omega_2 = (1+\sqrt{2})/\sqrt{2}$, then $\mu_1 = \mu_2 = -\mu_3 = 1$ is a solution of equation (3.4). This strongly contrasts with the two-dimensional case where accidental degeneracy can only occur when the frequencies are rationally related.

In § 4, we shall search for symmetry groups explaining the various types of accidental degeneracies discussed in the present section.

4. Symmetry group of the three-dimensional anisotropic oscillator

4.1. The case of three rational frequency ratios

When the oscillator frequencies satisfy the relation

$$k_1\omega_1 = k_2\omega_2 = k_3\omega_3 = \omega \tag{4.1}$$

where the integers k_1, k_2, k_3 are relatively prime per couple, the Hamiltonian symmetry group can easily be obtained by extending to three dimensions the analysis carried out for two dimensions in § 2. The set of states (3.2) is divided into $k_1k_2k_3$ subsets characterised by a triplet of indices $\boldsymbol{\lambda} = (\lambda_1\lambda_2\lambda_3)$, defined by equation (2.17) where i now goes from 1 to 3. The states belonging to subset $\boldsymbol{\lambda}$ are denoted by

$$|n_1n_2n_3)_{\boldsymbol{\lambda}} = |n_1k_1+\lambda_1, n_2k_2+\lambda_2, n_3k_3+\lambda_3\rangle \tag{4.2}$$

and the corresponding eigenvalues by

$$E_n^{(\boldsymbol{\lambda})} = \left(n + \sum_{i=1}^{3} (\lambda_i+\tfrac{1}{2})k_i^{-1}\right)\omega \qquad n = \sum_{i=1}^{3} n_i. \tag{4.3}$$

Each subset $\boldsymbol{\lambda}$ can be put into one-to-one correspondence with the full set of eigenstates of a three-dimensional isotropic oscillator of frequency ω. Hence the Hamiltonian symmetry group is SU(3), and its generators are the traceless operators obtained from the operators C_{ij}, $i, j = 1, 2, 3$, whose definition is similar to equation (2.25).

When the oscillator frequencies satisfy equation (4.1) where k_1, k_2, k_3 are not relatively prime per couple, some additional degeneracies appear between states belonging to different subsets $\boldsymbol{\lambda}$. This is better illustrated by a simple example. If $k_1 = 1$, $k_2 = k_3 = 2$, then there are four subsets of states, respectively characterised by $\boldsymbol{\lambda} = (000)$, (001), (010) and (011). The eigenstates corresponding to a given n value and belonging to subset (000) (respectively (001)) are degenerate with those corresponding to $n-1$ (respectively n) and belonging to subset (011) (respectively (010)). Hence there are two families of degenerate states, each made of two subsets, [(000), (011)] and [(001), (010)]. In general, denoting by K the least common multiple of k_1, k_2, k_3, and defining $\kappa_i = K/k_i$, $i = 1, 2, 3$, we can write equation (4.3) as follows:

$$E_n^{(\boldsymbol{\lambda})} = \left(n + K^{-1}\sum_{i=1}^{3} (\lambda_i+\tfrac{1}{2})\kappa_i\right)\omega = \left[n+p+K^{-1}\left(q+\tfrac{1}{2}\sum_{i=1}^{3}\kappa_i\right)\right] \tag{4.4}$$

where

$$\sum_{i=1}^{3} \lambda_i\kappa_i = pK+q \qquad q = 0, 1, \ldots, K-1. \tag{4.5}$$

We therefore obtain K families of degenerate states, each characterised by a given q value and made of $l=k_1k_2k_3/K$ subsets $\boldsymbol{\lambda}$.

It is obvious that in such a case, the above defined SU(3) group cannot connect all the degenerate eigenstates of the Hamiltonian, hence it is not its symmetry group. On the other hand, it is easy to determine the total degeneracy of the energy levels taking into account that for any subset $\boldsymbol{\lambda}$ the degeneracy of the level characterised by a given n is $(n+1)(n+2)/2$. The result is a quadratic polynomial in n, whose highest power term is $ln^2/2$. Such degeneracies are not the dimensions of the irreducible representations of any group except when $l=2$. In the latter case, Cisneros and McIntosh (1970) already noted that the degeneracies $(n+1)^2$ and $(n+1)(n+2)$ of the $q=0$ and 1 family levels respectively correspond to the dimensions of the irreducible representations $(n,0)$ and $(n+\frac{1}{2},\frac{1}{2})$ of SO(4), or strictly speaking of the irreducible representations $j_1=j_2=n/2$ and $j_1=(n+1)/2$, $j_2=n/2$ of its universal covering group SU(2)×SU(2). They also pointed out that the Ravenhall *et al* (1967) analysis of the isotropic oscillator with an impenetrable wall across the origin might be extended to such a case. In the remainder of this subsection, we shall carry out this extension along the lines of Louck *et al* (1973a).

Let us begin with the simplest $l=2$ case, corresponding to $k_1=1$, $k_2=k_3=2$, for which

$$E_{\nu_1\nu_2\nu_3}=[\nu_1+\tfrac{1}{2}(\nu_2+\nu_3)+1]\omega. \tag{4.6}$$

The two families respectively characterised by $q=0$ and $q=1$ span two subspaces $S^{(0)}$ and $S^{(1)}$ of Hilbert space. If we set

$$\nu_1=n-J+\tfrac{1}{2}q \qquad \nu_2=J+M \qquad \nu_3=J-M \tag{4.7}$$

where $J=q/2,\ 1+q/2,\dots,n+q/2$, and $M=-J,\ -J+1,\dots,J$, then the $(n+1)\times(n+1+q)$ eigenstates

$$|(n+\tfrac{1}{2}q,\tfrac{1}{2}q)JM)=|n-J+\tfrac{1}{2}q,J+M,J-M\rangle \tag{4.8}$$

belonging to $S^{(q)}$ and corresponding to a given n value, are degenerate since equation (4.6) now reads

$$E_n^{(q)}=(n+1+q/2)\omega. \tag{4.9}$$

Denoting by J_{1i}, J_{2i}, $i=1,2,3$, the SU(2)×SU(2) generators, and by $J_i=J_{1i}+J_{2i}$, $K_i=J_{1i}-J_{2i}$, $i=1,2,3$, those of the locally isomorphic SO(4) group, we immediately obtain from equation (4.8) that

$$J_+=J_1+iJ_2=\eta_2\xi_3 \qquad J_-=J_1-iJ_2=\eta_3\xi_2 \qquad J_0=J_3=\tfrac{1}{2}(\eta_2\xi_2-\eta_3\xi_3). \tag{4.10}$$

The remaining SO(4) generators can be written as

$$\begin{aligned}
K_+^{(q)}&=K_1^{(q)}+iK_2^{(q)}=-\alpha_q(N,\mathcal{J})\eta_2^2\xi_1+\eta_1\xi_3^2\alpha_q(N,\mathcal{J})+\beta_q(N,\mathcal{J})J_+\\
K_-^{(q)}&=K_1^{(q)}-iK_2^{(q)}=\alpha_q(N,\mathcal{J})\eta_3^2\xi_1-\eta_1\xi_2^2\alpha_q(N,\mathcal{J})+\beta_q(N,\mathcal{J})J_-\\
K_0^{(q)}&=K_3^{(q)}=\alpha_q(N,\mathcal{J})\eta_2\eta_3\xi_1+\eta_1\xi_2\xi_3\alpha_q(N,\mathcal{J})+\beta_q(N,\mathcal{J})J_0
\end{aligned} \tag{4.11}$$

where

$$\alpha_q(N,\mathcal{J})=(N+\mathcal{J}+1)^{1/2}[(2\mathcal{J}-1+q)(2\mathcal{J}+1-q)]^{-1/2} \tag{4.12}$$

$$\beta_q(N,\mathcal{J})=q(N+1)[2\mathcal{J}(\mathcal{J}+1)]^{-1} \tag{4.13}$$

and

$$N=\eta_1\xi_1+\tfrac{1}{2}(\eta_2\xi_2+\eta_3\xi_3) \qquad \mathcal{J}=(J^2+\tfrac{1}{4})^{1/2}-\tfrac{1}{2}. \tag{4.14}$$

The coefficients $\alpha_q(N, \mathcal{J})$ and $\beta_q(N, \mathcal{J})$ in equation (4.11) have been determined in order that the operators $K_i^{(q)}$ have the known SO(4) generator matrix elements (Biedenharn 1961) when taken between two states (4.8) corresponding to given n and q values. In each of the subspaces $S^{(0)}$ and $S^{(1)}$, the corresponding SO(4) algebra connects all the degenerate eigenstates. Hence a construction similar to that given in equation (2.25) leads to the generators J_i, K_i, $i = 1, 2, 3$, of the total Hamiltonian SO(4) symmetry group.

It is now straightforward to show that the remaining $l = 2$ cases can be reduced to the one discussed above. Without loss of generality, we may assume that the common factor 2 is contained in k_2 and k_3, and set $k_1 = k_1'$, $k_2 = 2k_2'$, $k_3 = 2k_3'$, where the integers k_1', k_2', and k_3' are relatively prime per couple. Then equation (3.3) reads

$$E_{\nu_1\nu_2\nu_3} = \left(k_1'^{-1}\nu_1 + \tfrac{1}{2}(k_2'^{-1}\nu_2 + k_3'^{-1}\nu_3) + \tfrac{1}{2}\sum_{i=1}^{3} k_i^{-1} \right)\omega \tag{4.15}$$

and there are $2k_1'k_2'k_3'$ families of degenerate states, each made of two subsets $\boldsymbol{\lambda}$. Instead of dividing the set of states (3.2) into $4k_1'k_2'k_3'$ subsets $\boldsymbol{\lambda}$ as in equation (4.2), let us first separate them into $k_1'k_2'k_3'$ subsets characterised by a triplet of indices $\boldsymbol{\lambda}' = (\lambda_1'\lambda_2'\lambda_3')$ defined by

$$\nu_i = \lambda_i' \bmod k_i' \qquad \lambda_i' = 0, 1, \ldots, k_i' - 1 \qquad i = 1, 2, 3. \tag{4.16}$$

The states belonging to a given subset $\boldsymbol{\lambda}'$ are labelled by three non-negative integers ν_1', ν_2', ν_3' as follows:

$$|\nu_1'\nu_2'\nu_3'\}_{\boldsymbol{\lambda}'} = |\nu_1'k_1' + \lambda_1', \nu_2'k_2' + \lambda_2', \nu_3'k_3' + \lambda_3'\rangle \tag{4.17}$$

and span a subspace $S^{(\boldsymbol{\lambda}')}$ of Hilbert space. The corresponding eigenvalues now read

$$E^{(\boldsymbol{\lambda}')}_{\nu_1'\nu_2'\nu_3'} = \left(\nu_1' + \tfrac{1}{2}(\nu_2' + \nu_3') + \sum_{i=1}^{3} (\lambda_i' + \tfrac{1}{2})k_i^{-1} \right)\omega \tag{4.18}$$

and only differ from equation (4.6) by an irrelevant constant. In $S^{(\boldsymbol{\lambda}')}$, new creation and annihilation operators $\bar{\eta}_i^{(\boldsymbol{\lambda}')}$, $\bar{\xi}_i^{(\boldsymbol{\lambda}')}$, $i = 1, 2, 3$, are defined by equation (2.20) where k_i and λ_i are replaced by k_i' and λ_i' respectively. Each subspace $S^{(\boldsymbol{\lambda}')}$ is then divided into the direct sum of two subspaces $S^{(\boldsymbol{\lambda}'q)}$, $q = 0, 1$, whose states are defined by equation (4.8) with $|\nu_1'\nu_2'\nu_3'\}_{\boldsymbol{\lambda}'}$ substituted for $|\nu_1\nu_2\nu_3\rangle$. In each of the $2k_1'k_2'k_3'$ subspaces $S^{(\boldsymbol{\lambda}'q)}$, we can construct SO(4) generators $J_i^{(\boldsymbol{\lambda}'q)}$, $K_i^{(\boldsymbol{\lambda}'q)}$, $i = 1, 2, 3$, by replacing η_i, ξ_i by $\bar{\eta}_i^{(\boldsymbol{\lambda}')}$, $\bar{\xi}_i^{(\boldsymbol{\lambda}')}$ in equations (4.10)–(4.14). The total Hamiltonian symmetry group is finally obtained by a procedure similar to equation (2.25).

4.2. The case of only one rational frequency ratio

Without loss of generality, we may assume that

$$k_1\omega_1 = k_2\omega_2 = \xi\omega_3 = \omega \tag{4.19}$$

where k_1 and k_2 are two relatively prime integers, and ξ is irrational. For any fixed non-negative integer value of ν_3, the eigenstates $|\nu_1\nu_2\nu_3\rangle$ have the accidental degeneracy pattern of the two-dimensional oscillator with a rational frequency ratio $\omega_2/\omega_1 = k_1/k_2$. Hence the Hamiltonian has an SU(2) symmetry group whose generators can be obtained from an equation similar to equation (2.25).

4.3. The case of no rational frequency ratio

Let us assume that the frequencies are not rationally related, but satisfy equation (3.4) for some triplet of relatively prime integers $(\mu_1\mu_2\mu_3)$. Without loss of generality, we may suppose that μ_1 and μ_2 are positive and μ_3 negative. Equation (3.4) then reads

$$\mu_1\omega_1+\mu_2\omega_2=|\mu_3|\omega_3. \tag{4.20}$$

The simplest relation of this type is

$$\omega_1+\omega_2=\omega_3 \tag{4.21}$$

corresponding to $\mu_1=\mu_2=|\mu_3|=1$. We shall determine the Hamiltonian symmetry group in three steps: first reduce the study of the general case (4.20) to that of the special one (4.21), then find the Hamiltonian symmetry group in the latter case, and finally go back to the general case and determine the corresponding Hamiltonian symmetry group.

Let us first pass from the Hamiltonian (3.1) to an oscillator Hamiltonian whose frequencies ω'_i, $i=1,2,3$, satisfy the following relation:

$$\omega'_1+\omega'_2=\omega'_3 \qquad \text{where} \qquad \omega'_i=|\mu_i|\omega_i \qquad i=1,2,3. \tag{4.22}$$

For this purpose, let us divide the set of states (3.2) into $\mu_1\mu_2|\mu_3|$ subsets, characterised by a triplet of indices $\boldsymbol{\lambda}=(\lambda_1\lambda_2\lambda_3)$, defined by equation (2.17) where k_i is replaced by $|\mu_i|$ and i goes from 1 to 3. The states belonging to subset $\boldsymbol{\lambda}$ are labelled by three non-negative integers ν'_1, ν'_2, ν'_3, denoted by

$$|\nu'_1\nu'_2\nu'_3)_{\boldsymbol{\lambda}}=|\nu'_1\mu_1+\lambda_1,\ \nu'_2\mu_2+\lambda_2,\ \nu'_3|\mu_3|+\lambda_3\rangle. \tag{4.23}$$

Their eigenvalues are given by

$$E^{(\boldsymbol{\lambda})}_{\nu'_1\nu'_2\nu'_3}=\sum_{i=1}^{3}\nu'_i\omega'_i+\sum_{i=1}^{3}(\lambda_i+\tfrac{1}{2})\omega_i. \tag{4.24}$$

Under transformation (2.20) where k_i is replaced by $|\mu_i|$ and i goes from 1 to 3, the restriction $H^{(\boldsymbol{\lambda})}$ of H to the subspace $S^{(\boldsymbol{\lambda})}$ spanned by the states (4.23) is transformed into an oscillator of frequencies ω'_i, $i=1,2,3$, satisfying equation (4.22):

$$H^{(\boldsymbol{\lambda})}=\sum_{i=1}^{3}\bar{\eta}^{(\boldsymbol{\lambda})}_i\bar{\xi}^{(\boldsymbol{\lambda})}_i\omega'_i+\sum_{i=1}^{3}(\lambda_i+\tfrac{1}{2})\omega_i. \tag{4.25}$$

This completes the first step of our analysis.

Turning ourselves to the determination of the symmetry group of $H^{(\boldsymbol{\lambda})}$, we note that those members of the set (4.23), for which both $\nu'_1+\nu'_3$ and $\nu'_2+\nu'_3$ have given values, are degenerate with a degeneracy equal to $\min(\nu'_1+\nu'_3,\ \nu'_2+\nu'_3)+1$. Let us further divide the set (4.23) into an infinite number of subsets, each characterised by a given integer value of

$$\lambda'=\nu'_1-\nu'_2. \tag{4.26}$$

The states belonging to a given subset span a subspace $S^{(\boldsymbol{\Lambda})}$, where $\boldsymbol{\Lambda}=\boldsymbol{\lambda}\lambda'$, and are labelled by two non-negative integers n_1, n_2 as follows:

$$|n_1n_2\}_{\boldsymbol{\Lambda}}=|n_1+\tfrac{1}{2}(|\lambda'|+\lambda'),\ n_1+\tfrac{1}{2}(|\lambda'|-\lambda'),\ n_2)_{\boldsymbol{\lambda}}. \tag{4.27}$$

Since their eigenvalues are

$$E^{(\boldsymbol{\Lambda})}_n=n(\omega'_1+\omega'_2)+c^{(\boldsymbol{\Lambda})} \qquad n=n_1+n_2 \tag{4.28}$$

where

$$c^{(\Lambda)}=\tfrac{1}{2}(|\lambda'|+\lambda')\omega_1'+\tfrac{1}{2}(|\lambda'|-\lambda')\omega_2'+\sum_{i=1}^{3}(\lambda_i+\tfrac{1}{2})\omega_i \tag{4.29}$$

the states (4.27) can be put into one-to-one correspondence with the eigenstates of a two-dimensional isotropic oscillator of frequency $\omega_1'+\omega_2'$.

The restriction $H^{(\Lambda)}$ of $H^{(\Lambda)}$ can be transformed into such an oscillator

$$H^{(\Lambda)}=\left(\sum_{i=1}^{2}\hat{\eta}_i^{(\Lambda)}\hat{\xi}_i^{(\Lambda)}\right)(\omega_1'+\omega_2')+c^{(\Lambda)} \tag{4.30}$$

by introducing in $S^{(\Lambda)}$ new creation and annihilation operators $\hat{\eta}_i^{(\Lambda)}$, $\hat{\xi}_i^{(\Lambda)}$, $i=1,2$, defined by

$$\begin{aligned}
\hat{\eta}_1^{(\Lambda)}&=\sqrt{2}(\bar{\eta}_1^{(\Lambda)}\bar{\xi}_1^{(\Lambda)}+\bar{\eta}_2^{(\Lambda)}\bar{\xi}_2^{(\Lambda)}+|\lambda'|)^{-1/2}\bar{\eta}_1^{(\Lambda)}\bar{\eta}_2^{(\Lambda)}\\
\hat{\eta}_2^{(\Lambda)}&=\bar{\eta}_3^{(\Lambda)}\\
\hat{\xi}_1^{(\Lambda)}&=\sqrt{2}(\bar{\eta}_1^{(\Lambda)}\bar{\xi}_1^{(\Lambda)}+\bar{\eta}_2^{(\Lambda)}\bar{\xi}_2^{(\Lambda)}+|\lambda'|+2)^{-1/2}\bar{\xi}_1^{(\Lambda)}\bar{\xi}_2^{(\Lambda)}\\
\hat{\xi}_2^{(\Lambda)}&=\bar{\xi}_3^{(\Lambda)}.
\end{aligned} \tag{4.31}$$

It is straightforward to check that their action on the states (4.27) is given by

$$\begin{aligned}
\hat{\eta}_1^{(\Lambda)}|n_1n_2\}_\Lambda&=(n_1+1)^{1/2}|n_1+1\ n_2\}_\Lambda\\
\hat{\xi}_1^{(\Lambda)}|n_1n_2\}_\Lambda&=n_1^{1/2}|n_1-1\ n_2\}_\Lambda
\end{aligned} \tag{4.32}$$

and similar relations for $\hat{\eta}_2^{(\Lambda)}$ and $\hat{\xi}_2^{(\Lambda)}$; hence, in $S^{(\Lambda)}$, they satisfy boson commutation relations of the type (2.12). The symmetry group of $H^{(\Lambda)}$ is therefore an SU(2) group, whose generators are the traceless operators obtained from

$$C_{ij}^{(\Lambda)}=\hat{\eta}_i^{(\Lambda)}\hat{\xi}_j^{(\Lambda)} \qquad i,j=1,2. \tag{4.33}$$

Finally going back to the original Hamiltonian H, by an argument similar to that used in § 2, we conclude that it has an SU(2) symmetry group, generated by the traceless operators obtained from

$$C_{ij}=\sum_{\Lambda}C_{ij}^{(\Lambda)}P^{(\Lambda)} \tag{4.34}$$

where $P^{(\Lambda)}$ is the projection operator onto $S^{(\Lambda)}$. The existence of such an SU(2) symmetry group for irrationally related frequencies satisfying equation (3.4) is a distinctive feature of the three-dimensional anisotropic oscillator as compared with the two-dimensional one. To our knowledge, it has not been reported so far in the literature.

5. Symmetry group of the three-dimensional oscillator in a strong magnetic field

To determine the symmetry group of the three-dimensional oscillator in a strong magnetic field in terms of the parameter α, and ultimately of the ratio β between the oscillator and cyclotron frequencies, it only remains to combine the analysis of § 4 with equations (2.29) and (2.10). The results are summarised in table 1.

In conclusion, for those β values for which accidental degeneracies appear in the spectrum, either the latter are due to an SU(2), an SU(3) or an SU(2) × SU(2) symmetry group, or they cannot be explained by the existence of any symmetry group. Whenever

Table 1. Symmetry group of the three-dimensional oscillator in a strong magnetic field in terms of the number of rational frequency ratios of the associated three-dimensional anisotropic oscillator. The frequencies of the latter are $\omega_1 = 1+\alpha$, $\omega_2 = 1-\alpha$, $\omega_3 = (1-\alpha^2)^{1/2}$, where $\alpha = (1+4\beta^2)^{-1/2}$ and $\beta = \Omega/\omega_c$. Here $l = k_1k_2k_3/K$, where K is the least common multiple of the three relatively prime integers k_1, k_2, k_3. In the case of zero rational frequency ratio, two examples are listed.

Rational frequency ratios	Conditions on α	Conditions on β	Symmetry group	Comments	
3	$\alpha, (1-\alpha^2)^{1/2} \in \boldsymbol{Q}$	$\beta, (1+4\beta^2)^{1/2} \in \boldsymbol{Q}$	SU(3)	$k_1\omega_1 = k_2\omega_2 = k_3\omega_3$	$l = 1$
			SU(2) × SU(2)	$k_1\omega_1 = k_2\omega_2 = k_3\omega_3$	$l = 2$
			—	$k_1\omega_1 = k_2\omega_2 = k_3\omega_3$	$l > 2$
1	$\alpha \in \boldsymbol{Q}, (1-\alpha^2)^{1/2} \notin \boldsymbol{Q}$	$\beta \notin \boldsymbol{Q}, (1+4\beta^2)^{1/2} \in \boldsymbol{Q}$	SU(2)		
0	some $\alpha \notin \boldsymbol{Q}$	some $\beta \in$ or $\notin \boldsymbol{Q}$	SU(2)	$\Sigma_i \mu_i\omega_i = 0$ $\mu_i \in \boldsymbol{Z}$	
	e.g. $\alpha \notin \boldsymbol{Q}$, $(1-\alpha^2)^{1/2} \in \boldsymbol{Q}$	$\beta, (1+4\beta^2)^{1/2} \notin \boldsymbol{Q}$, $\beta(1+4\beta^2)^{-1/2} \in \boldsymbol{Q}$		$\mu_2/\mu_1 = 1$ $\mu_3/\mu_1 = -2(1+4\beta^2)^{1/2}/\beta$	
	or $\alpha, (1-\alpha^2)^{1/2} \notin \boldsymbol{Q}$, $(1-\alpha^2)^{1/2}/\alpha \in \boldsymbol{Q}$	$\beta \in \boldsymbol{Q}, (1+4\beta^2)^{1/2} \notin \boldsymbol{Q}$		$\mu_2/\mu_1 = -1$ $\mu_3/\mu_1 = -1/\beta$	

β is close to one of such distinguished values, the non-degenerate eigenstates cluster into nearly degenerate multiplets, and we obtain an approximate symmetry group if an exact one is associated with the distinguished β value under consideration.

The classical limit of the quantum problem discussed in the present paper only retains part of the latter complexity. On the one hand, any classical anisotropic oscillator has an SU(3) symmetry group whatever the frequencies may be (Cisneros and McIntosh 1970). Hence, contrary to the quantum oscillator in a strong magnetic field, the classical one has an SU(3) symmetry group for any β value. On the other hand, the non-bijectiveness of the various canonical transformations used in the quantum picture, that is responsible for the complicated form of the symmetry group generators, has a counterpart in the classical picture, consisting in a Riemann sheet structure of phase space (Kramer *et al* 1978, Moshinsky and Seligman 1981).

Finally, it may be noted that the present analysis could be easily extended to an isotropic or anisotropic oscillator in crossed electric and magnetic fields (Schuh 1985).

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